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Wave drift force in a two-layer fluid of finite depth

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Abstract Based on the momentum and energy conservation principles, a compact calculation formula is analytically derived for the wave-drift force on a 2-D body floating in a two-layer fluid of finite depth. In a two-layer fluid, two different wave modes (the surface-wave mode with longer wavelength and the internal-wave mode with shorter wavelength) exist not only in the incident wave but also in the body-scattered wave, and these wave characteristics are properly incorporated in the obtained formula. It is noted that, unlike the single-layer case, the wave-drift force can be negative in the incident wave of surface-wave mode, if the transmitted wave with internal-wave mode is large. Numerical computations are implemented for a Lewis-form body by means of the boundary-integral-equation method with Green's function for the two-layer fluid problem. The effects of density ratio, interface position, and body motions on the wave-drift force are studied, and some important features are found for two-layer fluids.

Keywords Two-layer fluid · Wave drift force · Surface-wave mode · Internal-wave mode · Finite water depth

1 Introduction

Hydrodynamic studies of a body floating in a two-layer fluid of finite depth have been conducted for the radiation and diffraction problems which are of first order with respect to the incident-wave amplitude ([1,2] and the references therein). However, to the author's knowledge, no study has been made on the second-order wave-drift force in a two-layer fluid. For the case of a single-layer fluid, it is well known that the coefficient of the reflection wave is directly connected with the wave-drift force and its calculation formula is established by the far-field method based on the momentum and energy-conservation principles.

For a two-layer fluid, however, the analyses look complicated, even for first-order problems. For example, in the diffraction problem, two different incident waves of surface-wave mode (with longer wavelength) and internal-wave mode (with shorter wavelength) must be considered separately for a prescribed frequency, and each incident wave will be scattered by a body into two different wave modes. Thus, the energy of the incident wave may be transferred from one mode to the other. Furthermore, when the body is oscillating

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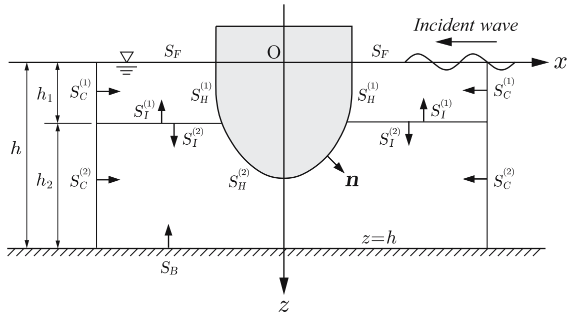

Fig. 1 Coordinate system and notations

in response to the incident wave, the body motion may change the reflected and transmitted waves. For this complicated wave field in a two-layer fluid, it is crucial to understand analytically what is the correct form of the calculation formula for the wave drift force, in what way two-layer effects are incorporated in the formula, and what are distinctive differences with respect to the single-layer case.

In this article, after the definition and formulation of the problem, the asymptotic expression of the velocity potential valid in the far field is obtained, in which the coefficients of reflected and transmitted waves are defined in terms of the Kochin functions for the radiation and diffraction problems in a two-layer fluid. Then on the basis of the momentum and energy conservation principles, analytical integrations in the far field and some mathematical transformations are performed to derive the desired calculation formula for the wave-drift force. The key to success in this procedure is to apply the orthogonality properties to the eigenfunctions in the two-layer fluid of finite depth.

Numerical computations are performed for a Lewis-form body, using the boundary-integral-equation method developed by Ten and Kashiwagi [1]. The density ratio and the interface position between the upper and lower layers are varied and those effects on the wave-drift force are studied. Furthermore, by showing the results for three cases where the body is completely fixed, only the heave motion is free, and all modes of body motion are free in response to incident waves, the effects of body motions on the wave drift force are also discussed. Lastly, some findings from the theoretical and numerical studies in this article are discussed in the Conclusions.

2 Mathematical formulation

We consider a 2-D floating body of general shape in a two-layer fluid of finite depth. The body may intersect the interface between the upper and lower layers and is assumed to oscillate sinusoidally in response to an incident wave with circular frequency ω. Figure 1 shows a Cartesian coordinate system and notations used in the analyses below, with the origin on the undisturbed free surface and the z-axis positive in the downward direction. The free surface, the interface, and the flat rigid bottom of the water are located at $z = 0$, $z = h_1$, and $z = h$, respectively.

Assuming both the upper and lower fluids to be incompressible and inviscid with irrotational motion, we can introduce the velocity potential in the form

$$\Phi^{(m)}(x,z,t) = \Re\mathfrak{e}\big[\,\phi^{(m)}(x,z)\mathrm{e}^{\mathrm{i}\omega t}\,\big], \quad m = 1,2, \tag{1}$$

$$\begin{aligned}\phi^{(m)}(x,z) &= \sum_{p=1}^{2} \frac{gA_p}{\mathrm{i}\omega}\,\phi_{Dp}^{(m)}(x,z) + \sum_{j=1}^{3} \mathrm{i}\omega X_j\,\phi_{Rj}^{(m)}(x,z) \\ &\equiv \sum_{p=1}^{2} \frac{gA_p}{\mathrm{i}\omega}\,\varphi_p^{(m)}(x,z),\end{aligned} \tag{2}$$

where

$$\varphi_p^{(m)}(x,z) = \phi_{Dp}^{(m)}(x,z) - K\sum_{j=1}^{3}\frac{X_j}{A_p}\,\phi_{Rj}^{(m)}(x,z), \tag{3}$$

$$\phi_{Dp}^{(m)}(x,z) = \phi_{Ip}^{(m)}(x,z) + \phi_{Sp}^{(m)}(x,z), \tag{4}$$

with $K = \omega^2/g$, and g being the gravitational acceleration.

Here the superscript (m) denotes the fluid layer, with $m = 1$ and 2 corresponding to the upper and lower layers, respectively. As described in [3], there can be two different wave modes in the incident wave in two-layer fluids for a prescribed frequency. Those modes are differentiated with subscript (p), and specifically $p = 1$ is referred to as the surface-wave mode and $p = 2$ as the internal-wave mode.

A_p in (2) denotes the amplitude of the incident wave at each mode. It is known that a simple relation holds at each mode on the amplitude ratio between the waves on the free surface and on the interface. However, the ratio between the waves of surface-wave and internal-wave modes on the free surface or the interface is not known a priori. Therefore, the diffraction problem must be solved for two different incident waves at a given frequency. In this article, A_p at each mode is defined in the theory as the incident wave on the free surface, whereas in numerical computations a larger amplitude is adopted as A_p at each wave mode (i.e., A_1 is the amplitude on the free surface and A_2 is the amplitude on the interface).

$\phi_{Dp}^{(m)}$ denotes the diffraction potential, which includes the incident-wave potential $\phi_{Ip}^{(m)}$ to be given as the input (explicit expressions of which will be shown below) and the scattering potential $\phi_{Sp}^{(m)}$. $\phi_{Rj}^{(m)}$ in (3) is the radiation potential with unit velocity in the jth direction ($j = 1$ for sway, $j = 2$ for heave, and $j = 3$ for roll) and the amplitude of the jth mode of motion, X_j/A_p, must be obtained by solving the equations of motions of a body, for which hydrodynamic forces must be computed with the solution of the boundary-value problem.

The governing equation for the velocity potentials is the 2-D Laplace equation

$$\frac{\partial^2\phi^{(m)}}{\partial x^2} + \frac{\partial^2\phi^{(m)}}{\partial z^2} = 0 \tag{5}$$

and the linearized boundary conditions to be satisfied on the free surface, the interface, and the rigid bottom of the lower layer are expressed as follows:

$$\frac{\partial\phi^{(1)}}{\partial z} + K\phi^{(1)} = 0 \quad \text{on } z = 0, \tag{6}$$

$$\left.\begin{aligned} &\frac{\partial\phi^{(1)}}{\partial z} = \frac{\partial\phi^{(2)}}{\partial z} \\ &\gamma\left(\frac{\partial\phi^{(1)}}{\partial z} + K\phi^{(1)}\right) = \frac{\partial\phi^{(2)}}{\partial z} + K\phi^{(2)} \end{aligned}\right\} \text{ on } z = h_1, \tag{7}$$

$$\frac{\partial\phi^{(2)}}{\partial z} = 0 \quad \text{on } z = h\,(= h_1 + h_2). \tag{8}$$

Here, by linearity, $\phi^{(m)}$ in the above can be any of the velocity potentials appearing in (1)–(4), and $\gamma = \rho_1/\rho_2 \leq 1$ is the density ratio, with ρ_m being the density of the upper ($m = 1$) and lower ($m = 2$) fluids.

Since the incident wave is independent of the presence of a body, the velocity potential of the incident wave, $\phi_{Ip}^{(m)}$, can be obtained from (5)–(8) and by specifying the amplitude of the incident wave on the free surface ($z = 0$) or the interface ($z = h_1$). As shown in Fig. 1, the incident wave is assumed to propagate from the positive x-axis. Then the velocity potential of the incident wave is expressed in the form

$$\phi_{Ip}^{(m)}(x,z) = Z^{(m)}(k_p;z)\,\mathrm{e}^{\mathrm{i}k_p x} \tag{9}$$

where

$$\left.\begin{aligned} Z^{(1)}(k;z) &= \frac{k\,\mathrm{ch}\,kz - K\,\mathrm{sh}\,kz}{k} \\ Z^{(2)}(k;z) &= \frac{K\,\mathrm{ch}\,kh_1 - k\,\mathrm{sh}\,kh_1}{k\,\mathrm{sh}\,kh_2}\,\mathrm{ch}\,k(z-h) \end{aligned}\right\} \tag{10}$$

and the variable k in (9) and (10) is the wavenumber satisfying the dispersion relation for a two-layer fluid given by

$$D(k) = K\big(k\,\mathrm{sh}\,kh - K\,\mathrm{ch}\,kh\big) + \varepsilon\big(K^2 - k^2\big)\,\mathrm{sh}\,kh_1\,\mathrm{sh}\,kh_2 = 0. \tag{11}$$

For brevity, the hyperbolic functions of $\cosh(x)$ and $\sinh(x)$ have been written as $\mathrm{ch}\,(x)$ and $\mathrm{sh}\,(x)$, respectively, and $\varepsilon = 1 - \gamma$ in (11); these notations will be used throughout in this article.

To determine the other velocity potentials associated with the disturbance by a body, the boundary condition on the body surface must be imposed, which can be given in the form

$$\left.\begin{aligned} \frac{\partial \phi_{Dp}^{(m)}}{\partial n} &= 0 \quad (p = 1,2) \\ \frac{\partial \phi_{Rj}^{(m)}}{\partial n} &= n_j \;(j = 1 \sim 3) \end{aligned}\right\} \text{ on } S_H^{(m)} \tag{12}$$

where n_j denotes the jth component $(n_1 = n_x, n_2 = n_z, n_3 = xn_z - zn_x)$ of the normal vector, which is defined as positive when directed into the fluid domain from boundaries (see Fig. 1).

The boundary-value problems for the disturbance velocity potentials (except for the incident-wave component) may be completed by imposing the radiation condition of generated waves radiating from the body.

3 Numerical solution method

The diffraction and radiation potentials formulated above are determined directly by the integral-equation method in terms of the Green function satisfying all homogeneous boundary conditions. This solution method can be applied to a general case where an arbitrary body intersects the interface between the upper and lower layers, and the derivation of the integral equation based on Green's theorem is shown in [1,2]. The results may be summarized in the form

$$C(\mathrm{P})\phi_\ell^{(m)}(\mathrm{P}) + \sum_{n=1}^{2}\int_{S_H^{(n)}} \phi_\ell^{(m)}(\mathrm{Q})\frac{\partial}{\partial n_{\mathrm{Q}}} G_n^{(m)}(\mathrm{P};\mathrm{Q})\,\mathrm{d}s$$

$$= \begin{cases} \phi_{Ip}^{(m)}(\mathrm{P}) & (\,\ell = Dp;\, p = 1,2\,) \\ \displaystyle\sum_{n=1}^{2}\int_{S_H^{(n)}} n_j(\mathrm{Q})\,G_n^{(m)}(\mathrm{P};\mathrm{Q})\,\mathrm{d}s & (\,\ell = Rj;\, j = 1 \sim 3\,) \end{cases} \tag{13}$$

where $\mathrm{P} = (x,z)$ and $\mathrm{Q} = (\xi,\zeta)$ denote the field and integration points, respectively, located on the body surface and $C(\mathrm{P})$ denotes the solid angle. $G_n^{(m)}(\mathrm{P};\mathrm{Q})$ represents the Green function, which has different forms depending on whether P and Q are in the upper or lower layer; details are shown in [1].

The so-called constant-panel collocation method is adopted for solving (13); that is, the body surface of $z > 0$ is divided into N segments and on each segment the unknown velocity potential is assumed to be constant. Then, considering N different points for $\mathrm{P}(x,z)$, we can recast (13) in a linear system of simultaneous equations for N unknowns.

In actual numerical computations, some additional field points are considered on both $z = 0$ and $z = h_1$ inside the body to remove the irregular frequencies. The resultant over-constrained simultaneous equations are solved using a least-squares method.

Once the velocity potentials on the body surface have been determined, it is straightforward to compute the hydrodynamic forces that must be used in solving the equations of motion of a body in each of the incident waves of surface-wave mode ($p = 1$) and internal-wave mode ($p = 2$). The calculation method for the motions of a body in waves is described in [2].

4 Velocity potentials in the far field

The analyses necessary for obtaining asymptotic expressions of the velocity potentials for $x \to \pm\infty$ may also be found in [1,2], and the results are summarized as follows:

$$\phi_{Dp}^{(m)} \sim \phi_{Ip}^{(m)} + \mathrm{i}\sum_{q=1}^{2} H_{Sp}^{\pm}(k_q)\, Z^{(m)}(k_q; z)\, \mathrm{e}^{\mp \mathrm{i} k_q x}, \tag{14}$$

$$\phi_{Rj}^{(m)} \sim \mathrm{i}\sum_{q=1}^{2} H_{Rj}^{\pm}(k_q)\, Z^{(m)}(k_q; z)\, \mathrm{e}^{\mp \mathrm{i} k_q x}, \tag{15}$$

where

$$H_{Sp}^{\pm}(k) = -\sum_{n=1}^{2} \int_{S_H^{(n)}} \phi_{Dp}^{(n)} \frac{\partial}{\partial n} \frac{W_n(k;\zeta)}{D'(k)}\, \mathrm{e}^{\pm \mathrm{i} k \xi}\, \mathrm{d}s, \tag{16}$$

$$H_{Rj}^{\pm}(k) = \sum_{n=1}^{2} \int_{S_H^{(n)}} \left\{ \frac{\partial \phi_{Rj}^{(n)}}{\partial n} - \phi_{Rj}^{(n)} \frac{\partial}{\partial n} \right\} \frac{W_n(k;\zeta)}{D'(k)}\, \mathrm{e}^{\pm \mathrm{i} k \xi}\, \mathrm{d}s, \tag{17}$$

$$\left.\begin{aligned} W_1(k;\zeta) &= \gamma\, \alpha(k) k\, \mathrm{sh}\, k h_2\, Z^{(1)}(k;\zeta) \\ W_2(k;\zeta) &= \alpha(k) k\, \mathrm{sh}\, k h_2\, Z^{(2)}(k;\zeta) \end{aligned}\right\} \tag{18}$$

$$\alpha(k) = \frac{K}{K\, \mathrm{ch}\, k h_1 - k\, \mathrm{sh}\, k h_1}, \tag{19}$$

$$\begin{aligned} D'(k) = \; & K(\mathrm{sh}\, kh + kh\, \mathrm{ch}\, kh - Kh\, \mathrm{sh}\, kh) \\ & + \varepsilon\big\{ -2k\, \mathrm{sh}\, kh_1\, \mathrm{sh}\, kh_2 + (K^2 - k^2)(h_1\, \mathrm{ch}\, kh_1\, \mathrm{sh}\, kh_2 + h_2\, \mathrm{sh}\, kh_1\, \mathrm{ch}\, kh_2) \big\}. \end{aligned} \tag{20}$$

Equations 16 and 17 are the Kochin functions (complex amplitude functions of the body-disturbance waves) computed from canonical velocity potentials in the diffraction and radiation problems. In terms of these Kochin functions and the complex amplitude of the jth mode of motion, X_j/A_p, to be obtained by solving the equations of motion of a body in the incident wave of the k_p-wave mode, the Kochin function representing the whole disturbance wave with wavenumber k_q can be obtained by linear superposition as follows:

$$H_p^{\pm}(k_q) \equiv H_{Sp}^{\pm}(k_q) - K \sum_{j=1}^{3} \frac{X_j}{A_p} H_{Rj}^{\pm}(k_q). \tag{21}$$

With this definition and taking the summation of both components of the surface-wave ($q = 1$) and internal-wave ($q = 2$) modes, the asymptotic expression of the velocity potential defined by (3) can be expressed in the following form:

$$\varphi_p^{(m)}(x,z) \sim Z^{(m)}(k_p;z)\,\mathrm{e}^{\mathrm{i}k_p x} + \sum_{q=1}^{2} R_{pq}\, Z^{(m)}(k_q;z)\,\mathrm{e}^{-\mathrm{i}k_q x} \quad \text{as } x \to +\infty, \tag{22}$$

$$\varphi_p^{(m)}(x,z) \sim \sum_{q=1}^{2} T_{pq}\, Z^{(m)}(k_q;z)\,\mathrm{e}^{\mathrm{i}k_q x} \quad \text{as } x \to -\infty, \tag{23}$$

where

$$\left.\begin{aligned} R_{pq} &= \mathrm{i}H_p^{+}(k_q) \\ T_{pq} &= \delta_{pq} + \mathrm{i}H_p^{-}(k_q) \end{aligned}\right\} \tag{24}$$

with δ_{pq} being the Kronecker delta.

R_{pq} and T_{pq} defined in (24) can be understood as the coefficients of reflected and transmitted waves, respectively, of the k_q-wave mode when the incident wave is of the k_p-wave mode.

5 Momentum-conservation principle

Following Maruo [4], a calculation formula for the wave-drift force in the horizontal direction can be derived on the basis of the momentum- and energy-conservation principles. Let us consider first the momentum-conservation principle in the x-axis in a two-layer fluid. With the same transformation as that for a single-layer fluid, the following equation may be obtained as a basis:

$$\sum_{m=1}^{2} \overline{\int_{S^{(m)}} \left\{ p^{(m)} n_x + \rho_m \frac{\partial \Phi^{(m)}}{\partial x} \left(\frac{\partial \Phi^{(m)}}{\partial n} - U_n \right) \right\} \mathrm{d}s} = 0, \tag{25}$$

where

$$\left.\begin{aligned} S^{(1)} &= S_H^{(1)} + S_C^{(1)} + S_I^{(1)} + S_F \\ S^{(2)} &= S_H^{(2)} + S_C^{(2)} + S_I^{(2)} + S_B \end{aligned}\right\}, \tag{26}$$

$$p^{(m)} = -\rho_m \left\{ \frac{\partial \Phi^{(m)}}{\partial t} + \frac{1}{2} \nabla \Phi^{(m)} \cdot \nabla \Phi^{(m)} \right\} + p_S^{(m)}, \tag{27}$$

$$\Phi^{(m)} = \Re\mathfrak{e}\left[\frac{gA_p}{\mathrm{i}\omega}\, \varphi_p^{(m)}(x,z)\, \mathrm{e}^{\mathrm{i}\omega t} \right]. \tag{28}$$

The overbar in (25) means the time-average over one period and U_n in (25) represents the normal velocity of the boundaries surrounding the fluid under consideration. $p_S^{(m)}$ in (27) denotes the static pressure independent of the disturbance velocity potential. As explicitly written in (28), only the incident wave of the k_p-wave mode ($p = 1$ or 2) is considered here.

As shown in Fig. 1, the control surface $S_C^{(m)}$ in the present study is parallel to the z-axis and in the linear theory the free surface S_F, the interface $S_I^{(m)}$, and the bottom of fluid S_B are parallel to the x-axis; these are fixed in space and thus $U_n = 0$ on these boundaries. On the other hand, the normal velocity of the body boundary must be equal to the normal velocity of the fluid and thus

$$U_n = \frac{\partial \Phi^{(m)}}{\partial n} \quad \text{on } S_H^{(m)}. \tag{29}$$

Taking these into consideration and retaining only quadratic terms in the velocity potential, we may write an expression for the wave-drift force acting in the negative direction of the x-axis as follows:

$$\begin{aligned}
F_D &\equiv \sum_{m=1}^{2} \overline{\int_{S_H^{(m)}} p^{(m)} n_x \, \mathrm{d}s} \\
&= \frac{1}{2}\rho_1 \left[\int_0^{h_1} \left\{ \overline{\frac{\partial \Phi^{(1)}}{\partial x}\frac{\partial \Phi^{(1)}}{\partial x} - \frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial \Phi^{(1)}}{\partial z}} \right\} \mathrm{d}z \right]_{-\infty}^{+\infty} \\
&\quad + \frac{1}{2}\rho_2 \left[\int_{h_1}^{h} \left\{ \overline{\frac{\partial \Phi^{(2)}}{\partial x}\frac{\partial \Phi^{(2)}}{\partial x} - \frac{\partial \Phi^{(2)}}{\partial z}\frac{\partial \Phi^{(2)}}{\partial z}} \right\} \mathrm{d}z \right]_{-\infty}^{+\infty} \\
&\quad - \rho_1 \int_{S_F} \overline{\frac{\partial \Phi^{(1)}}{\partial x}\frac{\partial \Phi^{(1)}}{\partial z}} \, \mathrm{d}x + \rho_1 \int_{S_I^{(1)}} \overline{\frac{\partial \Phi^{(1)}}{\partial x}\frac{\partial \Phi^{(1)}}{\partial z}} \, \mathrm{d}x - \rho_2 \int_{S_I^{(2)}} \overline{\frac{\partial \Phi^{(2)}}{\partial x}\frac{\partial \Phi^{(2)}}{\partial z}} \, \mathrm{d}x.
\end{aligned} \tag{30}$$

Here the square brackets with superscript $+\infty$ and subscript $-\infty$ in (30) means the difference between the quantities in the brackets evaluated at $x = +\infty$ and $x = -\infty$. The integrand in the integrals on S_F and S_I may be transformed in terms of the boundary conditions given by (6) and (7) as follows:
on S_F:

$$\frac{\partial \Phi^{(1)}}{\partial x}\frac{\partial \Phi^{(1)}}{\partial z} = -K\Phi^{(1)}\frac{\partial \Phi^{(1)}}{\partial x} = -\frac{1}{2}K\frac{\partial}{\partial x}\left\{\Phi^{(1)}\Phi^{(1)}\right\}, \tag{31}$$

on S_I:

$$\begin{aligned}
\rho_1 \frac{\partial \Phi^{(1)}}{\partial x}\frac{\partial \Phi^{(1)}}{\partial z} - \rho_2 \frac{\partial \Phi^{(2)}}{\partial x}\frac{\partial \Phi^{(2)}}{\partial z} &= \frac{\rho_1}{\gamma}\frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial}{\partial x}\left\{\gamma\,\Phi^{(1)} - \Phi^{(2)}\right\} \\
&= \frac{\rho_1}{\gamma}\frac{\partial \Phi^{(1)}}{\partial z}\frac{1-\gamma}{K}\frac{\partial}{\partial x}\frac{\partial \Phi^{(1)}}{\partial z} = \frac{1}{2}\rho_1\frac{1-\gamma}{\gamma K}\frac{\partial}{\partial x}\left\{\frac{\partial \Phi^{(1)}}{\partial z}\frac{\partial \Phi^{(1)}}{\partial z}\right\}.
\end{aligned} \tag{32}$$

For taking time average, the following formula may be useful

$$\overline{\Re\mathrm{e}\left[A\,\mathrm{e}^{\mathrm{i}\omega t}\right]\Re\mathrm{e}\left[B\,\mathrm{e}^{\mathrm{i}\omega t}\right]} = \frac{1}{2}\Re\mathrm{e}\left[A\,B^*\right], \tag{33}$$

where A and B are complex in general and the asterisk means the complex conjugate.
Substituting (31) and (32) in (30) and applying (33) with (28) gives the following result:

$$\begin{aligned}
F'_{Dp} &\equiv \frac{F_D}{\frac{1}{2}\rho_1 g A_p^2} \\
&= \frac{1}{2K}\left[\int_0^{h_1}\left\{\left|\frac{\partial \varphi_p^{(1)}}{\partial x}\right|^2 - \left|\frac{\partial \varphi_p^{(1)}}{\partial z}\right|^2\right\}\mathrm{d}z + \frac{1}{\gamma}\int_{h_1}^{h}\left\{\left|\frac{\partial \varphi_p^{(2)}}{\partial x}\right|^2 - \left|\frac{\partial \varphi_p^{(2)}}{\partial z}\right|^2\right\}\mathrm{d}z\right]_{-\infty}^{+\infty} \\
&\quad + \frac{1}{2}\left[\left|\varphi_p^{(1)}\right|_{z=0}^2\right]_{-\infty}^{+\infty} + \frac{1-\gamma}{2\gamma K^2}\left[\left|\frac{\partial \varphi_p^{(1)}}{\partial z}\right|_{z=h_1}^2\right]_{-\infty}^{+\infty}.
\end{aligned} \tag{34}$$

This may be regarded as an extension of the expression for a single-layer fluid to the case of a two-layer fluid, and, in fact, the last line in (34) can be understood as contributions from the square of the wave height at the free surface ($z = 0$) and the interface ($z = h_1$). However, this form is not convenient for analytical integration with respect to z, because the derivatives with respect to z are included. Thus, it is not straightforward to utilize the orthogonality properties of the eigenfunctions for a two-layer fluid as summarized in the Appendix.

To overcome this inconvenience, we consider a further transformation for the integrals including the derivatives with respect to z using the Laplace equation and the boundary conditions on $z = 0$, $z = h_1$, and $z = h$. Performing partial integrations and substituting (5)–(8), the following result can be justified:

$$\begin{aligned}\mathcal{I} &\equiv \int_0^{h_1} \frac{\partial \varphi^{(1)}}{\partial z}\frac{\partial \varphi^{(1)*}}{\partial z}\,\mathrm{d}z + \frac{1}{\gamma}\int_{h_1}^{h} \frac{\partial \varphi^{(2)}}{\partial z}\frac{\partial \varphi^{(2)*}}{\partial z}\,\mathrm{d}z \\ &= K\left\{ \left|\varphi^{(1)}\right|^2_{z=0} + \frac{1-\gamma}{\gamma K^2}\left|\frac{\partial \varphi^{(1)}}{\partial z}\right|^2_{z=h_1} \right\} \\ &\quad + \int_0^{h_1} \frac{\partial^2 \varphi^{(1)}}{\partial x^2}\varphi^{(1)*}\,\mathrm{d}z + \frac{1}{\gamma}\int_{h_1}^{h} \frac{\partial^2 \varphi^{(2)}}{\partial x^2}\varphi^{(2)*}\,\mathrm{d}z. \end{aligned} \tag{35}$$

Substituting this result in (34), we can see that the first line on the right-hand side of (35) cancels exactly the last terms in (34) to be evaluated at $z = 0$ and $z = h_1$.

Therefore, as a final result that is convenient for analytical integrations with respect to z, the following expression can be obtained:

$$F'_{Dp} = \frac{1}{2\gamma K}\left[\int_0^h w(z)\left\{ \left|\frac{\partial \varphi_p}{\partial x}\right|^2 - \frac{\partial^2 \varphi_p}{\partial x^2}\varphi_p^* \right\}\mathrm{d}z \right]_{-\infty}^{+\infty} \tag{36}$$

where $w(z)$ and φ_p are defined as

$$\begin{cases} w(z) = \gamma,\ \varphi_p = \varphi_p^{(1)} \text{ for } 0 \le z \le h_1, \\ w(z) = 1,\ \varphi_p = \varphi_p^{(2)} \text{ for } h_1 \le z \le h. \end{cases} \tag{37}$$

6 Energy-conservation principle

A relation to be obtained from the energy-conservation principle is usually used to derive a compact formula for the wave-drift force and also to check the accuracy of the computed results. With the same notations as for (25)–(28), a basis equation for the two-layer fluid may be given in the form

$$\overline{\sum_{m=1}^{2} \int_{S^{(m)}} \left\{ \rho_m \frac{\partial \Phi^{(m)}}{\partial t}\frac{\partial \Phi^{(m)}}{\partial n} - \left(p^{(m)} + \rho_m \frac{\partial \Phi^{(m)}}{\partial t}\right) U_n \right\} \mathrm{d}s} = 0. \tag{38}$$

With this equation, considering the normal velocity U_n of the boundaries, the work done by a body onto the fluid may be evaluated as follows:

$$\begin{aligned} W &\equiv \sum_{m=1}^{2} \overline{\int_{S_H^{(m)}} p^{(m)} U_n \,\mathrm{d}s} \\ &= -\rho_1 \left[\int_0^{h_1} \overline{\frac{\partial \Phi^{(1)}}{\partial t}\frac{\partial \Phi^{(1)}}{\partial x}}\,\mathrm{d}z \right]_{-\infty}^{+\infty} - \rho_2 \left[\int_{h_1}^{h} \overline{\frac{\partial \Phi^{(2)}}{\partial t}\frac{\partial \Phi^{(2)}}{\partial x}}\,\mathrm{d}z \right]_{-\infty}^{+\infty} \\ &\quad + \rho_1 \int_{S_F} \overline{\frac{\partial \Phi^{(1)}}{\partial t}\frac{\partial \Phi^{(1)}}{\partial z}}\,\mathrm{d}x - \rho_1 \int_{S_I^{(1)}} \overline{\frac{\partial \Phi^{(1)}}{\partial t}\frac{\partial \Phi^{(1)}}{\partial z}}\,\mathrm{d}x + \rho_2 \int_{S_I^{(2)}} \overline{\frac{\partial \Phi^{(2)}}{\partial t}\frac{\partial \Phi^{(2)}}{\partial z}}\,\mathrm{d}x. \end{aligned} \tag{39}$$

Substituting (28) and taking the time-average over one period using the formula (33), we may obtain the following result:

$$W'_p \equiv \frac{W}{\frac{1}{2}\rho_1 g A_p^2\left(\frac{\omega}{K}\right)}$$
$$= -\,\mathfrak{Im}\Big[\int_0^{h_1} \frac{\partial\varphi_p^{(1)}}{\partial x}\,\varphi_p^{(1)*}\,\mathrm{d}z + \frac{1}{\gamma}\int_{h_1}^{h} \frac{\partial\varphi_p^{(2)}}{\partial x}\,\varphi_p^{(2)*}\,\mathrm{d}z\Big]_{-\infty}^{+\infty}$$
$$+\,\mathfrak{Im}\int_{S_F} \frac{\partial\varphi_p^{(1)}}{\partial z}\,\varphi_p^{(1)*}\,\mathrm{d}x + \frac{1}{\gamma}\,\mathfrak{Im}\int_{S_I}\Big\{\frac{\partial\varphi_p^{(2)}}{\partial z}\,\varphi_p^{(2)*} - \gamma\,\frac{\partial\varphi_p^{(1)}}{\partial z}\,\varphi_p^{(1)*}\Big\}\,\mathrm{d}x. \tag{40}$$

In the above $\mathfrak{Im}$ means that only the imaginary part is to be taken.

Taking account of the boundary conditions on S_F and S_I as we did in deriving (31) and (32), one can easily prove that the integrals on S_F and S_I have no contributions because the integrands are real quantities. Therefore, with the notations of (37), the result can be written in the form

$$W'_p = -\frac{1}{\gamma}\,\mathfrak{Im}\Big[\int_0^{h} w(z)\,\frac{\partial\varphi_p}{\partial x}\,\varphi_p^*\,\mathrm{d}z\Big]_{-\infty}^{+\infty}. \tag{41}$$

Here it is noteworthy that the work done by a body must be zero when the body is fixed (as in the diffraction problem) or freely oscillating in waves without external oscillation devices supplying the energy.

7 Wave-drift force

Having prepared all necessary equations, let us derive the formula for the wave drift in a two-layer fluid. The asymptotic expressions of $\varphi_p^{(m)}$, given by (22) and (23), must be substituted in (36). To perform this procedure in general, Eq. 22 for instance can be written as

$$\varphi_p^{(m)}(x,z) = Z^{(m)}(k_p;z)\Big\{\mathrm{e}^{\mathrm{i}k_p x} + R_{pp}\,\mathrm{e}^{-\mathrm{i}k_p x}\Big\} + Z^{(m)}(k_q;z)\,R_{pq}\,\mathrm{e}^{-\mathrm{i}k_q x}, \tag{42}$$

with the convention that $p \neq q$; that is, when $p = 1$ (the incident wave is of the surface-wave mode) then $q = 2$, and when $p = 2$ (the incident wave is of the internal-wave mode) we have $q = 1$.

It should also be noted that, owing to the orthogonality followed in the Appendix, there is no need to consider cross terms between the k_p-wave and the k_q-wave in evaluating the integrals with respect to z. Therefore, using (42), we may write the integrand at $x = +\infty$ in (36) as follows:

$$\left|\frac{\partial\varphi_p^{(m)}}{\partial x}\right|^2 - \frac{\partial^2\varphi_p^{(m)}}{\partial x^2}\,\varphi_p^{(m)*} = 2\big\{Z^{(m)}(k_p;z)\big\}^2 k_p^2\left(1+|R_{pp}|^2\right) + 2\big\{Z^{(m)}(k_q;z)\big\}^2 k_q^2\,|R_{pq}|^2. \tag{43}$$

Here the integrals with respect to z can be identified with the normalization integral, whose explicit form is provided in the Appendix as follows:

$$\mathcal{F}(k) \equiv \frac{2k}{\gamma}\int_0^h w(z)\big\{Z(k;z)\big\}^2\,\mathrm{d}z = \frac{K}{k} + kh\,\frac{(K\,\mathrm{ch}\,kh_1 - k\,\mathrm{sh}\,kh_1)^2}{\gamma\,k^2\,\mathrm{sh}^2 kh_2}$$
$$+\frac{\varepsilon}{\gamma}\frac{h_1}{k}\Big[\Big(1-\frac{k^2}{K^2}+\frac{1}{Kh_1}\Big)\big(K\,\mathrm{ch}\,kh_1 - k\,\mathrm{sh}\,kh_1\big)^2 + \gamma\,\frac{(K^2-k^2)^2}{K^2}\,\mathrm{sh}^2 kh_1\Big]. \tag{44}$$

With these results, the integral at $x = +\infty$ in (36) takes the following form:

$$\big[\,F'_{Dp}\,\big]_{+\infty} = \frac{1}{2K}\Big\{k_p\left(1+|R_{pp}|^2\right)\mathcal{F}(k_p) + k_q\,|R_{pq}|^2\,\mathcal{F}(k_q)\Big\}. \tag{45}$$

In the same manner, the integral at $x = -\infty$ can be evaluated analytically. Namely, with convention of $p \neq q$, (23) is written as

$$\varphi_p^{(m)}(x,z) = T_{pp}\,Z^{(m)}(k_p;z)\,\mathrm{e}^{\mathrm{i}k_p x} + T_{pq}\,Z^{(m)}(k_q;z)\,\mathrm{e}^{\mathrm{i}k_q x} \tag{46}$$

and then it follows that

$$\left|\frac{\partial \varphi_p^{(m)}}{\partial x}\right|^2 - \frac{\partial^2 \varphi_p^{(m)}}{\partial x^2}\,\varphi_p^{(m)*} = 2\big\{Z^{(m)}(k_p;z)\big\}^2 k_p^2 \big|T_{pp}\big|^2 + 2\big\{Z^{(m)}(k_q;z)\big\}^2 k_q^2 \big|T_{pq}\big|^2. \tag{47}$$

Therefore, in terms of (44), the result of the integral at $x = -\infty$ takes the form

$$\big[\,F'_{Dp}\,\big]_{-\infty} = \frac{1}{2K}\Big\{\,k_p\big|T_{pp}\big|^2\mathcal{F}(k_p) + k_q\big|T_{pq}\big|^2\mathcal{F}(k_q)\,\Big\}. \tag{48}$$

The wave-drift force must be given by the difference between (45) and (48). Therefore, the result from (36) is expressed as

$$F'_{Dp} = \frac{1}{2K}\Big[\,k_p\left(1 + \big|R_{pp}\big|^2 - \big|T_{pp}\big|^2\right)\mathcal{F}(k_p) + k_q\left(\big|R_{pq}\big|^2 - \big|T_{pq}\big|^2\right)\mathcal{F}(k_q)\,\Big]. \tag{49}$$

As the next step, let us consider a relation to be obtained from the energy-conservation principle given by (41). With the same convention for $\varphi_p^{(m)}$ and the orthogonality properties for the integrals with respect to z, the final result of (41) can be expressed as follows:

$$W'_p = -\frac{1}{2}\,\Big\{\,\left(1 - \big|R_{pp}\big|^2 - \big|T_{pp}\big|^2\right)\mathcal{F}(k_p) - \left(\big|R_{pq}\big|^2 + \big|T_{pq}\big|^2\right)\mathcal{F}(k_q)\Big\}. \tag{50}$$

As noted at the end of the preceding section we have $W'_p = 0$ for the case where a body is fixed or freely oscillating in the incident wave. Therefore, the energy-conservation principle takes the form

$$\left(1 - \big|R_{pp}\big|^2 - \big|T_{pp}\big|^2\right)\mathcal{F}(k_p) = \left(\big|R_{pq}\big|^2 + \big|T_{pq}\big|^2\right)\mathcal{F}(k_q). \tag{51}$$

Substitution of this in (49) gives the final form of the calculation formula for the wave-drift force in a two-layer fluid:

$$F'_{Dp} = \frac{1}{K}\Big[\,k_p\big|R_{pp}\big|^2\mathcal{F}(k_p) + \Big\{\frac{k_p + k_q}{2}\big|R_{pq}\big|^2 + \frac{k_p - k_q}{2}\big|T_{pq}\big|^2\Big\}\mathcal{F}(k_q)\,\Big]. \tag{52}$$

By considering the limiting case of $\gamma \to 1$, let us confirm the corresponding formula for a single-layer fluid. For $\gamma = 1$, $k_2 \to \infty$ and the internal wave no longer exists, and hence

$$p = 1,\ \mathcal{F}(k_2) = 0,\ k_1 = k,\ K = k\tanh kh. \tag{53}$$

In this limiting case, $\varepsilon = 0$ in (44); thus, the coefficient associated with the normalization integral can be transformed into

$$\mathcal{J} \equiv \frac{k}{K}\,\mathcal{F}(k) = 1 + \frac{h}{K}\,\frac{(K\,\mathrm{ch}\,kh_1 - k\,\mathrm{sh}\,kh_1)^2}{\mathrm{sh}^2 kh_2} = 1 + \frac{h}{K}(K\,\mathrm{sh}\,kh - k\,\mathrm{ch}\,kh)^2 = 1 + \frac{2kh}{\mathrm{sh}\,2kh}. \tag{54}$$

Therefore, it follows from (52) that

$$F'_D = \frac{F_D}{\frac{1}{2}\rho g A^2} = \big|R\big|^2\left\{1 + \frac{2kh}{\mathrm{sh}\,2kh}\right\}, \tag{55}$$

with R being the coefficient of the reflected wave in a single-layer fluid. This result is well known as a formula for the wave-drift force in water of finite depth.

Another noteworthy aspect of (52) is the possibility that the wave-drift force in a two-layer fluid be negative. The wave-drift force to be computed from (52) is mostly positive. In particular for the case of $p = 2$ (i.e., for the incident wave of the internal-wave mode), the value of (52) is definitely positive because $k_2 > k_1$. However, for the case of $p = 1$, the value of (52) can be negative if the value of $|T_{12}|$ (the transmitted wave with wavenumber k_2 in the incident wave of the surface-wave mode) is relatively large.

When the energy is not conserved owing to viscous effects such as viscous damping in roll, relation (51) obtained from the energy-conservation principle cannot be used. However, even in this case, the momentum-conservation principle holds and thus the wave-drift force can be computed with (49).

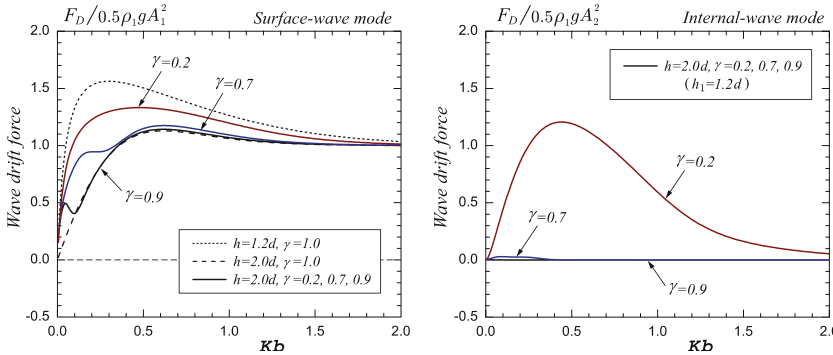

Fig. 2 Wave-drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids: effect of the difference in the fluid density for the case where all body motions are fixed

8 Numerical results and discussions

Numerical computations were performed for a Lewis-form body as used in the previous study of first-order radiation [1] and diffraction [2] problems. This Lewis-form body can be represented by a conformal mapping with two nondimensional parameters; those are the half-breadth to draft ratio, $H_0 = b/d = 0.833$, and the sectional area ratio, $\sigma = A/Bd = 0.9$ (in real dimensions, the breadth $B = 2b = 0.2\,\text{m}$ and the draft $d = 0.12\,\text{m}$). Since this body is symmetrical with respect to x, only half of the body surface was discretized into 40 segments for all computations in this article. With this number of segments, satisfaction of the energy-conservation principle given by (51) was virtually perfect with the order of error being 10^{-4} for both cases of body motions fixed and free to oscillate in waves.

8.1 Effects of the density ratio

To see the effects of the density ratio on the second-order wave-drift force, computations were implemented for the same parameters as those in the study of the first-order radiation and diffraction problems; that is, $\gamma = 1.0$, 0.9, 0.7, and 0.2 with the depths of the fluid layers fixed at $h_1 = 1.2d$ and $h = 2.0d$. As $\gamma \to 1$, the fluid reduces to a single-layer fluid of $h = 2.0d$. Conversely as $\gamma \to 0$, the lower fluid behaves more like a rigid block, and the results are expected to approach those for a single-layer fluid with upper-layer depth $h_1 = 1.2d$. To illustrate this behavior, computations were also carried out for single-layer fluids of $h = 1.2d$ and $2.0d$.

Figure 2 shows the nondimensional value of the wave-drift force for the case where all body motions are fixed, in which the left-hand and right-hand sides are for the surface-wave and internal-wave modes, respectively, of the incident wave.

It should be noted first that the wave-drift force shown in Fig. 2 is always positive over the whole range of frequency, although theoretically there is a possibility of negative value in the incident wave of the surface-wave mode. The results for $\gamma = 0.9$ are close to those for a single-layer fluid of $h = 2.0d$, except for very low frequencies, and the results for $\gamma = 0.7$ are also almost the same as those for $\gamma = 0.9$ in the frequency range of $Kb > 0.4$. On the other hand, for $\gamma = 0.2$, the results in the incident wave of surface-wave mode tend to approach the results for a single-layer fluid of $h = 1.2d$, and the nondimensional value of the drift force in the incident wave of the internal-wave mode also becomes large. (Note that the amplitude A_2 of the incident wave of the internal-wave mode is taken as that on the interface.)

Figure 3 shows computed results for the same parameters but for the case where only the heave motion is free to oscillate. Compared to Fig. 2, a big difference can be seen in lower frequencies, where the drift force in two-layer fluids becomes negative in the incident wave of the surface-wave mode over a certain

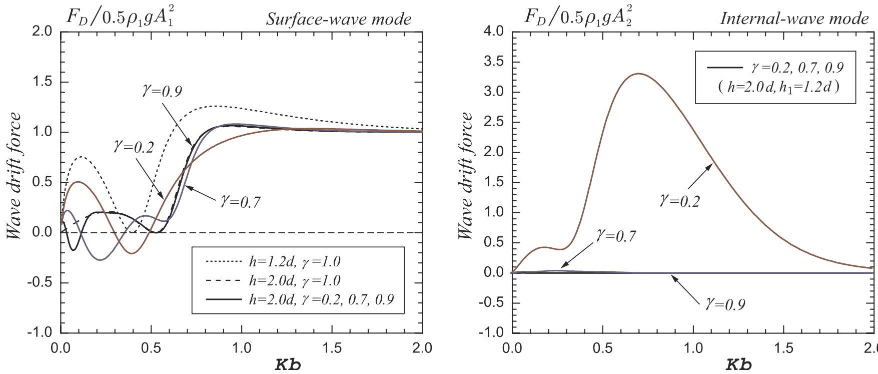

Fig. 3 Wave-drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids: effect of the difference in the fluid density for the case where only the heave motion is free to oscillate

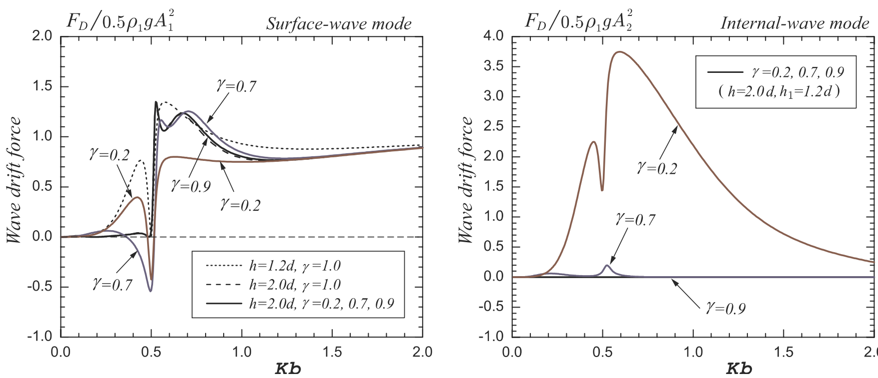

Fig. 4 Wave-drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids: effect of the difference in the fluid density for the case where all body motions are free to oscillate

frequency range, which can be attributed to a larger value of T_{12} in the calculation formula of (52). It can be said that the results for $\gamma = 0.7$ are almost the same as those for $\gamma = 0.9$ and for a single-layer fluid of $h = 2.0d$ at frequencies of $Kb > 0.6$ which is higher than the resonant frequency in heave. The wave-drift force in the incident wave of the internal-wave mode is always positive but very small for larger values of γ. However, for $\gamma = 0.2$, the nondimensional value becomes larger than double the corresponding value in the diffraction case, owing to the effect of heave motion.

Figure 4 shows the results when all modes (heave, sway, and roll) of body motion are free to oscillate. In the present computations, the gyrational radius in roll is set to $\kappa_{xx} = 0.6b$ and the vertical distance between the center of gravity and the free surface is set to $\overline{\mathrm{OG}} = 0.45b$. A rapid change can be seen around $Kb \simeq 0.5$ both for the surface-wave and internal-wave modes, which is obviously due to the resonance in roll. (It should be noted that the roll amplitude near resonance is unrealistically very large because the viscous damping is not considered in the present theory.) When the incident wave is of surface-wave mode, the wave-drift forces for $\gamma = 0.2$ and 0.7 become negative at frequencies lower than the roll resonant frequency. However, a marked difference when compared to Fig. 3 is that the wave-drift force is almost zero at very low frequencies. Another thing to be noted is that the results for $\gamma = 0.9$ are very close to those for a single-layer fluid over the whole frequency range, including the heave and roll resonant frequencies; which implies that a small difference in the fluid density between the upper and lower layers gives no prominent difference in the wave-drift force and motion characteristics.

The results shown above are for the case where a body floats in the upper fluid only and the interface is located at a relatively deeper position. The horizontal force (like the wave-drift force) may be affected

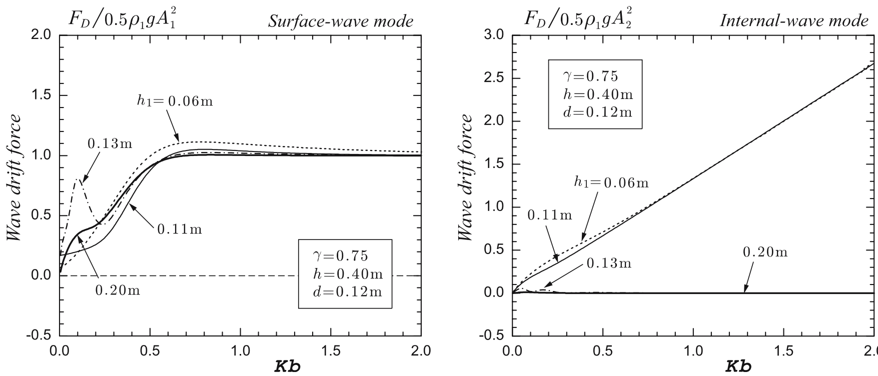

Fig. 5 Wave-drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids: effect of the interface position for the case where all body motions are fixed

by the presence of internal waves near a body, which is the case particularly when a body intersects the interface and this will be studied next.

8.2 Effects of the interface position

For the same Lewis-form body ($b = B/2 = 0.1$ m and $d = 0.12$ m) and fixed values of $h = 0.4$ m and $\gamma = 0.75$, only the vertical position of the interface was changed from $h_1 = 0.06 – 0.20$ m, including the case where the body intersects the interface.

Figure 5 shows computed results of the wave-drift force for the case where all body motions are fixed, and like before the left-hand and right-hand sides are for the surface-wave and internal-wave modes, respectively, of the incident wave. In the incident wave of the surface-wave mode, the drift force is always positive, and no prominent difference exists among the results for different interface positions, except that undulatory variation can be seen at $Kb < 0.25$ for the case of $h_1 = 0.13$ m where the interface is located just below the bottom of the body ($d = 0.12$ m). On the other hand, in the incident wave of the internal-wave mode, a remarkable change can be seen depending on whether a body intersects the interface. When the interface position is deeper than the draft of a body, the wave-drift force is negligibly small. However, once a body intersects the interface, the wave-drift force becomes large and increases almost linearly with respect to Kb, for which we may envisage that the internal incident wave will be blocked by a body and almost all waves may be reflected; that is, the coefficient of R_{22} in the calculation formula (52) is largely different depending on whether the body intersects the interface.

Figure 6 shows the results when only the heave motion is free to oscillate. Obviously, owing to the heave motion, the wave-drift force becomes small in frequencies lower than the heave resonant frequency, which implies that longer incident waves transmit because the heave is free to oscillate. It should be noted, however, that the drift force at $h_1 = 0.13$ m fluctuates in lower frequencies and becomes negative in a certain frequency range, which is due to the effect of waves with internal-wave mode, as can be conjectured from (52).

On the other hand, in the incident wave of the internal-wave mode, no marked difference can be seen as compared to Fig. 5, which implies that the reflection-wave coefficient R_{22} (especially when a body intersects the interface) is not much influenced by the heave motion and the hydrodynamic situation near the cross-point between body and interface may be viewed locally as a diffraction problem regardless of the heave motion.

Lastly, Fig. 7 shows results for various vertical positions of the interface when all modes of the body motion are free to oscillate. As shown from [2], the resonant frequency in roll changes slightly depending

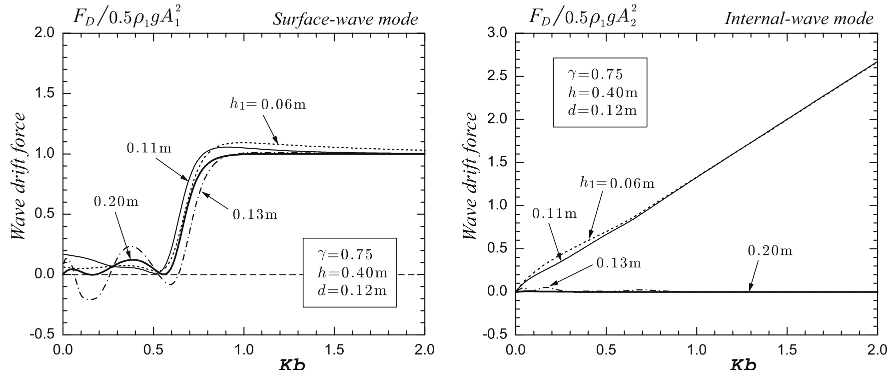

Fig. 6 Wave-drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids: effect of the interface position for the case where only the heave motion is free to oscillate

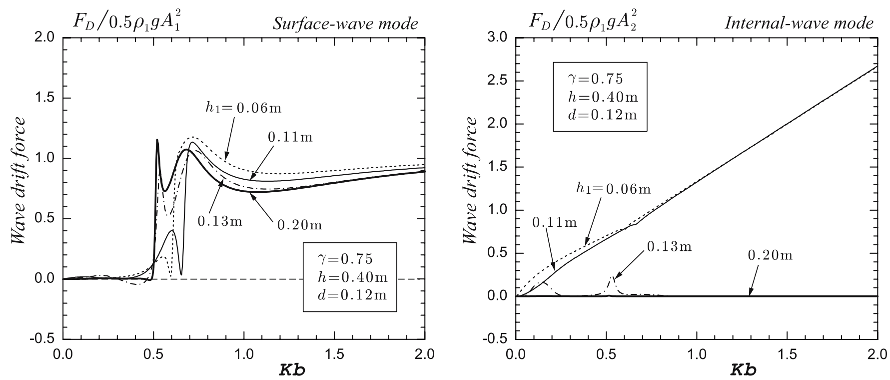

Fig. 7 Wave-drift force on a Lewis-form body ($H_0 = 0.833$, $\sigma = 0.9$) in two-layer fluids: effect of the interface position for the case where all body motions are free to oscillate

on the position of the interface, which is due mainly to the change in the roll restoring moment. Therefore, the frequency where rapid variation in the wave-drift force appears is slightly different depending on the vertical position of the interface. It can be seen that the wave-drift force is almost zero at lower frequencies, irrespective of the interface position. Another point to be emphasized is that the wave-drift force looks always positive when a body intersects the interface (at $h_1 = 0.11$ and 0.06 m), which means that the transmitted wave with internal-wave mode T_{12} in the calculation formula of (52) is relatively small. When the incident wave is of the internal-wave mode, as compared to Figs. 5 and 6, a slight difference can be seen around the roll resonance, but we should note that the roll amplitude near the resonance becomes unrealistically very large for lack of viscous damping in the present study.

9 Conclusions

The wave-drift force in a 2-D two-layer fluid of finite depth has been studied with the potential-flow assumption. Based on the momentum and energy-conservation principles, a compact calculation formula for the wave-drift force was obtained; the key to success was to use effectively the orthogonality relations for the eigenfunctions in a two-layer fluid. Owing to the presence of the interface, for a prescribed frequency, there can exist two different incident waves with surface-wave mode (longer wavelength) and internal-wave mode (shorter wavelength), and each incident wave will be diffracted by a body into two different wave modes and hence the energy of the incident wave may be transferred from one mode to the other. The wave-drift force in this rather complicated situation was described with only one equation, which includes the coefficients of reflected and transmitted waves in a two-layer fluid. An important feature to be seen from this calculation formula is that the possibility of negative drift force exists in the incident wave of the surface-wave mode; this can be exerted by a large value of the transmitted wave with internal-wave mode.

Numerical computations were performed with the boundary-integral-equation method using the Green function for the two-layer fluid problem. Computed results of the wave-drift force were shown for both incident waves with surface-wave and internal-wave modes and also for three cases where all body motions are completely fixed, only the heave motion being free, and all body motions being free to oscillate. Furthermore, by changing the density ratio and the interface position including the case where a body intersects the interface, those effects on the wave-drift force were discussed.

The main results obtained from the present numerical study may be summarized as follows:

(1) When the body motions are fixed, the wave-drift force appears to be positive for all frequencies, regardless of the density ratio and the interface position. However, when the position of the interface is slightly lower than the bottom of a body and the body motions are free to respond to the incident wave of a surface-wave mode, the wave-drift force becomes negative at frequencies lower than the resonant frequency of body motion.
(2) When the difference in the fluid density between the upper and lower layers is large (say $\gamma = 0.2$), the wave-drift force becomes large, even in the incident wave of an internal-wave mode. On the other hand, when the difference in the fluid density is small (say $\gamma = 0.9$), the results are very close to those for a single-layer fluid over the whole frequency range, except at very low frequencies when all or some of the body motions are fixed.
(3) When a body intersects the interface, the body reflects most of the incident wave of the internal-wave mode, particularly at higher frequencies, and hence the wave-drift force, nondimensionalized in terms of the square of the wave amplitude on the interface, seems to increase linearly in proportion to the square of the frequency.

Appendix

The orthogonality properties of the eigenfunctions with respect to z in a two-layer fluid problem are explained in [1], a summary of which is given below.

As explicitly given by (10), the z-dependent functions in the upper and lower layers are denoted by $Z^{(1)}(k;z)$ and $Z^{(2)}(k;z)$, respectively. If necessary, the eigenfunctions corresponding to the eigenvalues $k = k_p$ $(p = 1, 2)$ are represented by a subscript, e.g., $Z_p^{(1)}(k_p;z)$ and $Z_p^{(2)}(k_p;z)$.

The orthogonality can be proven in the same way as that in the Sturm–Liouville eigenvalue problem, and the basic equation is given as

$$\begin{aligned}\mathcal{L} &\equiv \gamma\int_0^{h_1}\left\{\frac{d^2Z_1^{(1)}}{dz^2}Z_2^{(1)} - Z_1^{(1)}\frac{d^2Z_2^{(1)}}{dz^2}\right\}dz + \int_{h_1}^{h}\left\{\frac{d^2Z_1^{(2)}}{dz^2}Z_2^{(2)} - Z_1^{(2)}\frac{d^2Z_2^{(2)}}{dz^2}\right\}dz \\ &= (k_1^2 - k_2^2)\left[\gamma\int_0^{h_1} Z^{(1)}(k_1;z)Z^{(1)}(k_2;z)\,dz + \int_{h_1}^{h} Z^{(2)}(k_1;z)Z^{(2)}(k_2;z)\,dz\right] \\ &\equiv (k_1^2 - k_2^2)\int_0^h w(z)Z(k_1;z)Z(k_2;z)\,dz,\end{aligned} \tag{56}$$

where

$$\begin{cases} w(z) = \gamma, \quad Z(k_p;z) = Z^{(1)}(k_p;z) \quad 0 \le z \le h_1 \\ w(z) = 1, \quad Z(k_p;z) = Z^{(2)}(k_p;z) \quad h_1 \le z \le h \end{cases} \tag{57}$$

Integrating by parts for the first line of (56) and substituting the boundary conditions on $z = 0$, $z = h_1$, and $z = h$ (these are the same in form as (6)–(8)), one can easily prove that $\mathcal{L} = 0$ for the case of $k_1 \neq k_2$; that is,

$$\int_0^h w(z) Z(k_1; z) Z(k_2; z)\, \mathrm{d}z = 0 \quad \text{for } k_1 \neq k_2. \tag{58}$$

This means that there is no need to consider the integrals of the cross-terms between the k_1-wave and k_2-wave modes.

Next, the normalization integral for the case of $k_1 = k_2 \equiv k$ can be obtained by taking the limit of $k_1 \to k_2$ in (56). Substituting $k_1 = k_2 + \delta k$ in (56), considering the Taylor expansion with respect to k, and retaining only the term of $O(\delta k)$, the desired result for the normalization integral can be derived and expressed in the form

$$\begin{aligned}
\mathcal{F} &\equiv \frac{2k}{\gamma} \int_0^h w(z) \big\{ Z(k; z) \big\}^2 \mathrm{d}z \\
&= \frac{K}{k} \Big[\big\{ Z^{(1)} \big\}^2 \Big]_{z=0} + \frac{kh}{\gamma} \Big[\big\{ Z^{(2)} \big\}^2 \Big]_{z=h} \\
&\quad + \varepsilon\, kh_1 \left[\big\{ Z^{(1)} \big\}^2 + \frac{1}{\gamma} \Big(1 + \frac{1}{Kh_1} - \varepsilon \frac{k^2}{K^2} \Big) \big\{ Z^{(1)\prime} \big\}^2 + 2 \frac{k}{K} Z^{(1)} Z^{(1)\prime} \right]_{z=h_1} \\
&= \frac{K}{k} + kh \frac{(K \operatorname{ch} kh_1 - k \operatorname{sh} kh_1)^2}{\gamma\, k^2 \operatorname{sh}^2 kh_2} \\
&\quad + \frac{\varepsilon}{\gamma} \frac{h_1}{k} \left[\Big(1 - \frac{k^2}{K^2} + \frac{1}{Kh_1} \Big) \big(K \operatorname{ch} kh_1 - k \operatorname{sh} kh_1 \big)^2 + \gamma \frac{(K^2 - k^2)^2}{K^2} \operatorname{sh}^2 kh_1 \right].
\end{aligned} \tag{59}$$

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